Optimal violation of Leggett-Garg inequality for arbitrary spin and emergence of classicality through unsharp measurement

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We consider temporal correlations of particles with arbitrary spin. We show that the Leggett-Garg inequality can be maximally violated irrespective of the value of spin, thus improving upon an earlier result by Kofler and Brukner [Phys. Rev. Lett. **99**, 180403 (2007)]. Our proof is accomplished through a suitable adoption of a measurement scheme which has previously been employed for studying the spatial correlation in a system with arbitrary spin. We next consider generalized or unsharp measurements as a method for coarse graining in a quantitative manner, and show that this inequality can not be violated below a precise value of the sharpness parameter. We then apply the Fine's theorem in context of the Leggett-Garg (LG) inequality in order to derive LG-CH inequalities which provide a sufficient condition for emergence of classicality.

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I. INTRODUCTION

Quantum physics has been enormously successful in explaining nonclassical manifestations of nature. Bell's theorem [1] proves through quantum violation of the Bell-CHSH [2] inequality, that correlations arising from spatially separated quantum systems cannot be explained if we adopt local realism- a genuine classical world view. Quantum mechanical violation of the Bell-CHSH inequality for dichotomic measurements is upper bounded by $2\sqrt{2}$ [3]. The question as to how quantum correlations are fundamentally limited has been explored as an optimization problem which can be done efficiently, and assured to reach the global optimum since it represents a so-called semidefinite program [4]. Upper bounds on the quantum violation of more general Bell-type inequalities have also been obtained [5].

On the other hand, from the very beginning of quantum mechanics it has been of great interest to understand how the quanutum-classical transition occurs. The classical limit problem arises if quantum mechanics is taken to be a universal theory encompassing classical physics. One might then expect the typically classical behavior of a macrosystem to emerge within a quantum mechanical description if the relevant quantities are large compared to the Planck's constant. However, it has been shown that quantum features persist in the limit of large quantum numbers such as the number of constituents of a system [6], or the value of its spin angular momentum [7–9]. In such cases the magnitude of violation of the relevant local realist inequalities generally seems to increase even in the limit of large numbers of particles and large spins considered simultaneously [10]. Nonclassicality of multiparty and multilevel systems in the context of spatial correlation have been extensively studied, belying expectations of classical properties emerging automatically for 'large' quantum systems [11, 12].

Besides the issue of local realism, Leggett and Garg in a seminal paper [13] have provided a way to test quantum mechanics in the macroscopic domain by deriving an inequality based on macroscopic realism- a classical world view dealing with temporal correlations. The two assumptions of macrorealism(MR) are, (i) *macrorealism per se:* a macroscopic object which has available to it two or more macroscopically distinct states is at any given time in a definite one of those states, and (ii) *noninvasive measurability (NIM):* it is possible in principle to determine which of these states the system is in without any effect on the state itself or on the subsequent system dynamics. Another assumption implicit in the theory is induction which states that system properties are not determined by the final conditions. Various studies have been performed recently on theoretical aspects of the Leggett-Garg inequality (LGI) and its generalizations [14–18], along with a couple insightful reviews [19]. Several experiments testing the violation of LGI have also been reported [20]. It has been of considerable interest to investigate the extent to which various Bell ienqualities are violated by quantum mechanics for different types of systems [21].

Within the context of temporal correlations, the question as to how the classical world emerges out of quantum physics has been discussed by Kofler and Brukner [15]. Inspired by the earlier ideas of Peres [22] on the classical limit of quantum mechanics, they have presented a different theoretical approach to macroscopic realism and classical physics within quantum theory. They showed that if consecutive eigenvalues of the spin component can be sufficiently resolved, the LGI will be violated for arbitrary large spin. On the other hand, with sufficiently coarse grained measurement, classical laws would emerge for a macroscopic system with very large dimension. This approach is rather different from the decoherence program. However, the violation they obtained for large spin is not maximal. It remains unclear as to why the violation is lesser that the value $2\sqrt{2}$ achieved for spin $1/2$ particles

and remains constant asymptotically for large spin. The choice of observables may indeed have a role to play in the quantum violation of the LGI.

In order to obtain optimal violation of the LGI for arbitrary spin, we employ a variant of the measurement scheme suggested earlier by Gisin and Peres [9] in the context of testing local realism. Interestingly, for the case of spatial correlations the above measurement scheme [9] yields maximal violation of a local realist inequality only for half-integral spin systems. For integral spin systems the amount of violation drops, and the value of $2\sqrt{2}$ is achieved only when the spin becomes infinitely large. It has earlier been shown [23] that the success probability for Hardy type protocols [24] is much higher in temporal correlations than in spatial ones. Our finding is thus indicative of another aspect of difference between spatial and temporal correlations in addition to the main results.

In the second part of our work we focus on the issue of quantum-classical transition by adopting unsharp measurement as a completely different method of coarse graining from the method followed in [15]. Unsharp measurement, a form of positive operator valued measurement (POVM), is well studied in the quantum formalism [25, 26]. Emergence of classicality is argued by transforming the LGI to a necessary and sufficient condition for macrorealism for dichotomic measurement at four different time following Fine's theorem [27]. We find that below a definite degree of sharpness of measurement, a macrorealist inequality like LGI can not be violated, and hence, there exists a non-invasive realist model compatible with classical physics for reproducing correlation statistics. Through this formalism we find a way to compare between different inequalities based on different assumptions, as also discussed in a recent work [28] involving one of us.

The main results obtained in this work are: (i) we obtain violation of LGI up to $2\sqrt{2}$ for arbitrary spin which is optimal for dichotomic measurements, (ii) by application of unsharp measurement we show that satisfaction of the LGI ensues below a specific value of the sharpness parameter, and (iii) application of the Fine's theorem in context of the LGI leads to LG-CH type of inequalities which provide a sufficient condition for MR. The paper is organized as follows. In the next section we briefly discuss the derivation of the LGI based on an ontological model [29], and provide a quick review of its quantum violation for arbitrary value of spin using the formalism of Ref. [15]. In Section III we employ another scheme of measurement [9] to show that optimal violation for arbitrary spin can be obtained. We further describe a process of conceptual experimental realization of the scheme. In Section IV we apply the formalism of unsharp measurement on the LGI, presenting a different method of coarse graining responsible for the quantum-to-classical transition. In Section V we apply Fine's theorem to temporal correlation and show how LGI becomes a sufficient condition for MR. Some concluding remarks are presented in Section V.

II. THE LEGGETT-GARG INEQUALITY

Following the ontological framework discussed in[29], we begin with a short derivation of the LGI. In this framework any Heisenberg picture operator in quantum mechanics can be written as an average over a set of hidden variables λ. The role of the initial state is to provide a probability distribution on the set of hidden variables, which we denote as $\rho(\lambda)$, called the ontic state. The average of an observable can be written as

$$< \hat{A}(t) >= \int d\lambda A(\lambda, t)\rho(\lambda), \tag{1}$$

where $A(\lambda, t)$ is the value taken by the observable on the hidden variable λ. The correlation between two observables is given by

$$< \hat{B}(t_2)\hat{A}(t_1) >= \int d\lambda B(\lambda, t_2)A(\lambda, t_1)\rho(\lambda|A, t_1). \tag{2}$$

Non-invasive measurability (NIM) can be defined as $\rho(\lambda|A, t_1, B, t_2...) = \rho(\lambda)$, i.e., a measurement performed does not change the distribution of λ (like the locality condition in Bell's theorem). Let us take A, B as observables measured on a single system at different times denoted by $Q(t_1), Q(t_2)$. Now, following similar steps as in the derivation of the Bell inequality, one obtains

$$\begin{aligned} & < \hat{Q}(t_2)\hat{Q}(t_1) > - < \hat{Q}(t_4)\hat{Q}(t_1) > \\ &= \int d\lambda[Q(\lambda, t_2)Q(\lambda, t_1) - Q(\lambda, t_4)Q(\lambda, t_1)]\rho(\lambda|Q, t_1) \\ &= \int d\lambda Q(\lambda, t_2)Q(\lambda, t_1)[1 \pm Q(\lambda, t_4)Q(\lambda, t_3)]\rho(\lambda|Q, t_1) \\ &- \int d\lambda Q(\lambda, t_4)Q(\lambda, t_1)[1 \pm Q(\lambda, t_3)Q(\lambda, t_2)]\rho(\lambda|Q, t_1). \end{aligned} \tag{3}$$

Now,

$$\begin{aligned} | < \hat{Q}(t_2)\hat{Q}(t_1) > - < \hat{Q}(t_4)\hat{Q}(t_1) > | \leq 2 \pm \\ [\int d\lambda Q(\lambda, t_4)Q(\lambda, t_3)\rho(\lambda|Q, t_1) \\ + \int d\lambda Q(\lambda, t_3)Q(\lambda, t_2)\rho(\lambda|Q, t_1)]. \end{aligned} \tag{4}$$

Invoking NIM, we have,

$$\begin{aligned} | < \hat{Q}(t_2)\hat{Q}(t_1) > - < \hat{Q}(t_4)\hat{Q}(t_1) > | \mp \\ [< \hat{Q}(t_3)\hat{Q}(t_2) > + < \hat{Q}(t_4)\hat{Q}(t_3) >] \leq 2. \end{aligned} \tag{5}$$

This is four term Leggett-Garg inequality.

In an actual experiment, $Q(t)$, a macrovariable measured at time t, is found to take a value $+1(-1)$ depending on whether the system is in the state 1(2).

We consider series of measurements with the same initial conditions such that in the first series Q is measured at times t_1 and t_2, in the second at t_2 and t_3, in the third at t_3 and t_4, and in the fourth at t_1 and t_4 (here $t_1 < t_2 < t_3 < t_4$). From such measurements one obtains the temporal correlations $C_{ij} = \langle Q_i Q_j \rangle = p^{++}(Q_i, Q_j) - p^{-+}(Q_i, Q_j) - p^{+-}(Q_i, Q_j) + p^{--}(Q_i, Q_j)$, where $p^{++}(Q_i, Q_j)$ is the joint probability of getting '+' outcomes at both times t_i and t_j. Experimentally, these joint probabilities are determined from the Bayes' rule $p^{++}(Q_i, Q_j) = p^{+}(Q_i)p^{+|+}(Q_j|Q_i)$, where $p^{+|+}(Q_j|Q_i)$ is the conditional probability of getting '+' outcome at t_j given that '+' outcome occurs at t_i.

Let us now briefly describe how quantum violation of the LGI was obtained in[15]. Consider precession of a spin 1/2 particle under the unitary evolution $U_t = e^{-i\omega t \sigma_x/2}$, where ω is the angular precession frequency. Measurement of σ_z at times t_1 and t_2 yields the temporal correlation $C_{12} = \cos\omega(t_2 - t_1)$. Here the state transformation rule is given by $\rho \to P_\pm \rho P_\pm / Tr[P_\pm \rho P_\pm]$. Choosing equidistant measurement times with time difference $\Delta t = t_2 - t_1 = \pi/4\omega$, the maximum value taken by the l.h.s of Eq.(6) is given by $2\sqrt{2}$. For a spin j system with a maximally mixed initial state $\frac{1}{2j+1}\sum_{m=-j}^{m=+j} |m\rangle\langle m|$, evolving unitarily under $U_t = e^{-i\omega t \hat{J}_x}$, measurement of the dichotomic parity operator $\sum_{m=-j}^{m=+j}(-1)^{j-m}|m\rangle\langle m|$, leads to the two-time correlation function given by

$$C_{12} = \sin[(2j+1)\omega\Delta t]/(2j+1)\sin[\omega\Delta t]. \tag{6}$$

With these correlations the LGI expressed as $K = C_{12} + C_{23} + C_{34} - C_{14} \leq 2$ becomes

$$K = \frac{3\sin x}{x} - \frac{\sin 3x}{3x} \leq 2, \tag{7}$$

where $x = (2j+1)\omega\Delta t$. For $x \approx 1.054$, the maximal violation in this case is obtained for infinitely large j, with the value 2.481, i.e., 42 percent short of the largest violation of $2\sqrt{2}$ allowed by quantum theory.

III. OPTIMAL VIOLATION FOR ARBITRARY SPIN

We now show how the maximum correlation up to $2\sqrt{2}$ which is the upper bound of quantum theory for dichotomic measurements, can be achieved not only for spin 1/2 particles, but for systems having arbitrary spin too.

Lemma: If a dichotomic observable Q is measured successively at times t_i and t_j on any state ρ of a two dimensional system evolving unitarily, then the two-time correlation function is given by $C_{ij} = \frac{1}{2}tr[Q(t_i)Q(t_j)]$. Here $Q(t_i) = U^\dagger(t_i)QU(t_i)$ and $Q(t_j) = U^\dagger(t_j)QU(t_j)$ are time evolved observables in the Heisenberg picture.

Proof: The initial state ρ is evolved to $U(t_i)\rho U^\dagger(t_i)$. At t_i, Q is measured and according to the outcome '$\pm$', the post-measurement state becomes $P_\pm U(t_i)\rho U^\dagger(t_i)P_\pm / tr[P_\pm U(t_i)\rho U^\dagger(t_i)P_\pm]$, where $P_\pm$ are the two orthogonal projectors of the observable Q, and $P_\pm U(t_i)\rho U^\dagger(t_i)P_\pm/p_\pm = P_\pm$. Here, $p_\pm = tr[P_\pm U(t_i)\rho U^\dagger(t_i)]$, are probability of getting outcomes '$\pm$'. Again, this post-measurement state is evolved to time t_j and becomes $U(\Delta t)P_\pm U^\dagger(\Delta t)$, with $\Delta t = t_j - t_i$. Now, the conditional probabilities are given by $p_{k|l} = tr[P_k U(\Delta t)P_l U^\dagger(\Delta t)$, where $p_{k|l}$ denotes the probability of getting an outcome k at time t_j when the outcome l occurs at time t_i. Hence, the two-time correlation is given by

$$\begin{aligned} C_{ij} &= p_+(p_{+|+} - p_{-|+}) + p_-(p_{-|-} - p_{+|-}) \\ &= p_+(tr[(P_+ - P_-)U(\Delta t)P_+U^\dagger(\Delta t)]) \\ &\quad + p_-(tr[(P_- - P_+)U(\Delta t)P_-U^\dagger(\Delta t)]) \end{aligned} \tag{8}$$

Now, using $P_+ - P_- = Q$, $tr[Q] = 0$, $p_+ + p_- = 1$, and the cyclic property of the trace, we have $C_{ij} = tr[P_+U^\dagger(\Delta t)QU(\Delta t)]$. Since, $P_+ = \frac{\mathbb{I}+Q}{2}$ and $U(\Delta t) = U(t_2)U^\dagger(t_1)$, we finally have

$$\begin{aligned} C_{ij} &= tr[QU^\dagger(\Delta t)QU(\Delta t)]/2 \\ &= tr[U^\dagger(t_1)QU(t_1)U^\dagger(t_2)QU(t_2)]/2. \end{aligned} \tag{9}$$

This completes the proof of the lemma.

Theorem: For any state of a single quantum system of arbitrary spin there exists observables with eigenvalues ± 1 and a measurement scheme such that the Leggett-Garg inequality can be maximally violated.

Proof: Let $\Gamma_x, \Gamma_y, \Gamma_z$ be block-diagonal matrices, in which each block is an ordinary Pauli matrix, σ_x, σ_y and σ_z respectively. The only nonvanishing elements are given by

$$\begin{aligned} (\Gamma_x)_{2n-1,2n} = (\Gamma_x)_{2n,2n-1} &= 1 \\ (\Gamma_y)_{2n-1,2n} = i, (\Gamma_y)_{2n,2n-1} &= -i \\ (\Gamma_z)_{n,n} &= (-1)^{n-1} \end{aligned} \tag{10}$$

Suppose mixed states of spin j particles coming from a source are in diagonal form in some basis $\{|k\rangle\}$, i.e.,

$$\sum_{k=-j}^{k=j} p_k |k\rangle\langle k| = \sum_{k=1/2(0)}^{k=j} (p_k|+k\rangle\langle +k| + p_{-k}|-k\rangle\langle -k|) \tag{11}$$

where, $\sum_{k=-j}^{k=j} p_k = 1$. We define an observable Q following [9] in the way given below:

$$\begin{aligned} Q &= \frac{\Gamma_z + \Pi}{\sqrt{2j+1}} \\ &= (\sigma_z^1 \oplus \sigma_z^2 + ... \oplus \sigma_z^j + \Pi)/\sqrt{2j+1} \end{aligned} \tag{12}$$

where, Π is the null matrix when $N(=2j+1)$ is even, and for odd N the only nonvanishing element of Π is $(\Pi)_{N,N} = \frac{1}{\sqrt{2}}$. In order to maintain optimal violation of the four-term LGI, we require our time-evolved observables to remain in the block diagonal form mentioned above. This can not be ensured by arbitrary rotations of the SG apparatus in space, except for two dimensional systems. However, this is achieved if each block is evolved separately [9]. As $\oplus_j \exp^{i\theta_j\sigma_x}\,\sigma_z \exp^{-i\theta_j\sigma_x} = \oplus_j \exp^{i\theta_j\sigma_x} \oplus_j \sigma_z^j \oplus_j \exp^{-i\theta_j\sigma_x}$, time evolution of the system is affected by

$$U(t) = \exp^{-i\theta_1\sigma_x} \oplus \exp^{-i\theta_2\sigma_x} \oplus ... \oplus \exp^{-i\theta_j\sigma_x} \quad (13)$$

We explain in next paragraph explicitly how this kind of evolution and measurements are realised experimentally. First, the system is evolved to time t_1 and Q is measured. The post-measurement state is further evolved to time t_2, and Q is measured again. This scheme can be recast into the Heisenberg picture. Taking all $\theta_j = \omega t/2 = \alpha$, we have

$$U^\dagger(t)QU(t) = (\cos\alpha\Gamma_z + \sin\alpha\Gamma_y + \Pi)/\sqrt{2j+1}. \quad (14)$$

The two-time correlation function C_{12} using lemma 1, for even N is given by

$$\begin{aligned} C_{12} &= Tr[U^\dagger(t_1)QU(t_1)U^\dagger(t_2)QU(t_2)] \\ &= [\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2]. \end{aligned} \quad (15)$$

For odd N, one gets

$$C_{12} = \frac{2j(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2) + 1/\sqrt{2}}{(2j+1)}. \quad (16)$$

Similarly, C_{23} and C_{34} are also obtained. For obtaining the correlation function C_{14}, the operator Q is taken to be $(\Gamma_z - \Pi)/\sqrt{2j+1}$. Now, in order to obtain the maximal violation of the LGI, we choose the time intervals such that $\alpha_1 = 0, \alpha_2 = \pi/4, \alpha_3 = \pi/2, \alpha_4 = 3\pi/4$. Hence, the value of the Leggett-Garg sum for the spin j system is given by $2\sqrt{2}$. ■

We now outline a conceptual scheme for verifying the correlations (15,16) following[9]. Such a scheme has been experimentally implemented for spin-1 entangled particles [12]. For the present purpose, consider spin j particles emerging from an initial ensemble. These particles are assumed to possess not only a magnetic moment (an interaction energy $\mu B_z J_z$), but also an electric quadrupole moment (an interaction energy proportional to $E_z J_z^2$). The particles first pass through inhomogeneous electric fields producing beams with $|m_z| = j, j-1...0(1/2)$. Taken separately, each of these beams with given $|m_z|$ are passed through a uniform magnetic field B_z producing energy difference, E, between them. Then an rf pulse generates Rabi oscillations of frequency $\omega = E/\hbar$ among them. This captures the unitary evolution given by (13). After evolving for a time t_1, σ_z is measured. Then, each post-measured beam is again evolved, and at a time t_2, σ_z is measured. This same procedure is done many times varying the time of measurements randomly. Correlation statistics are calculated from the measured data. For half-integral spin (N even) this procedure is exact. For integral spin (odd N), there is an unpaired beam corresponding to $m_z = 0$ for which special treatment is needed to get maximum violation. The beam corresponding to $m_z = 0$ is evolved by $U(t) = e^{-i\theta j_y}$, where, j_y is the y-th component of the corresponding spin operator j. For example, for a spin 1 system, the evolution leads to

$$e^{-i\theta_1 j_y}|m_z = 0\rangle = \frac{\sin\theta_1}{\sqrt{2}}|-1\rangle + \cos\theta_1|0\rangle - \frac{\sin\theta_1}{\sqrt{2}}|1\rangle \quad (17)$$

Next, measurement is performed by an inhomogeneous electric field, which is a two-outcome measurement characterised by the projectors $P_+ = P_1 + P_{-1}$ and $P_- = P_0$. Here, $P_{\pm 1}$ is the projector on the subspace spanned by $m_z = |\pm 1\rangle$ components and P_0 is projector on the $|0\rangle$ state. According to the outcome, the post-measured states are $(|-1\rangle - |1\rangle)/\sqrt{2}$ with probability $p_+ = \sin^2\theta_1$, and $|0\rangle$ with probability $p_- = \cos^2\theta_1$, respectively. These post-measurement states are again evolved to time t_2. The '+' outcome states evolve to $(\cos\theta_2|-1\rangle - \sqrt{2}\sin\theta_2|0\rangle - \cos\theta_2|1\rangle)/\sqrt{2}$, and the '$-$' outcome states evolve as given by Eq.(17). The conditional probabilities are $p_{+|+} = \cos^2\theta_2, p_{-|+} = \sin^2\theta_2, p_{+|-} = \sin^2\theta_2, p_{-|-} = \cos^2\theta_2$, where $\theta_2 = \omega\Delta t$. The two-time correlation function is $\cos(2\omega\Delta t)$, which is the same as in the qubit case, and hence gives the maximum violation up to $2\sqrt{2}$.

IV. DISAPPEARANCE OF VIOLATION THROUGH UNSHARP MEASUREMENT

The purpose of this section is to show that with sufficiently unsharp measurements, the outcome statistics can be described by classical theory. Using a particular form of coarse graining it was shown in Ref. [15] that when the resolution of the apparatus is much greater than the intrinsic quantum uncertainty, i.e., $\Delta m >> \sqrt{j}$, the outcomes appear to obey classical laws. Under this condition the measurements become fuzzy enough for the noninvasive assumption to become essentially valid and the system dynamics mimics the rotation of a classical spin coherent state. However, in this formalism there does not exist any sharp cut-off for the value of the apparatus resolution beyond which classicality emerges. Or, in other words, given a specific quantum system with a particular intrinsic uncertainty, it is not clear as to what is the

precise value of the apparatus resolution above which the condition of coarse graining is satisfied.

Here we follow the theory of unsharp observables [25] which as an element of unsharp reality provides the necessary ingredient in modelling of the emergence of classical behaviour within quantum mechanics in a precise and quantitative manner. It has been shown by Kar and Roy [30] that for the value of the sharpness parameter $\lambda \leq 1/2^{1/4}$, the CHSH inequality is always satisfied for spin 1/2 systems. As the spin or polarization observables of entangled particles in an EPR experiment are measured with progressively more limited accuracy, there is a corresponding progressive degradation of the degree of Bell violation. Violation of the Bell inequality becomes unobservable above a certain degree of inaccuracy [26]. Unsharp measurement for a class of LGIs has been considered for two level systems in a recent work [28] . Here we extended this approach for arbitrary spin systems.

Let us first describe briefly the formalism of unsharp measurements [25] relevant to our present analysis. In projector valued measurements the observables are self-adjoint operators having projectors as spectral, i.e., $A \equiv \{P_i | \sum P_i = \mathbb{I}, P_i^2 = P_i\}$. The probability of getting the i-th outcome is $tr[\rho P_i]$ for the state ρ. Extending to positive operator valued measures (POVM), the observables are self-adjoint operators but with spectral as positive operators within the interval $[0, \mathbb{I}]$, i.e., $E \equiv \{E_i | \sum E_i = \mathbb{I}, 0 < E \leq \mathbb{I}\}$. Similarly, the probability of getting the i-th outcome is $tr[\rho E_i]$. Effects ($E_i s$) represent quantum events that may occur as outcomes of a measurement. A subclass of effects of particular interest are the regular effects, characterized by the property that their spectrum extends both above and below the value 1/2. For two outcome measurements this notion is captured by the effect, $E_\lambda = (\mathbb{I} + \lambda n_i \sigma_i)/2, i = 1, 2, 3.$, with $\lambda \in (0, 1]$. Thus, the set of effects can be written as a linear combination of sharp projectors with white noise, $E_\lambda \equiv \{E_+^\lambda, E_-^\lambda | E_+^\lambda + E_-^\lambda = \mathbb{I}\}$, given by

$$E_\pm^\lambda = \frac{1+\lambda}{2} P_\pm + \frac{1-\lambda}{2} P_\mp = \lambda P_\pm + \frac{1-\lambda}{2}\mathbb{I}. \tag{18}$$

This can be thought of as projectors becoming noisy reflecting inaccuracy of the experiment. Hence, the sharpness or dialation parameter λ can be estimated from the difference between the really observable data and that predicted by sharp observables. Under this unsharp measurement, the state transformation for the maximally mixed initial state is given by the generalised Lüdders operation

$$\begin{aligned}\rho_\pm^{PM}(t_1) &= \sqrt{E_{\pm\lambda}}\rho\sqrt{E_{\pm\lambda}}/tr[\sqrt{E_{\pm\lambda}}\rho\sqrt{E_{\pm\lambda}}] \\ &= \sqrt{\frac{1}{2}(\mathbb{I} \pm \lambda\sigma_z)}\frac{\mathbb{I}}{2}\sqrt{\frac{1}{2}(\mathbb{I} \pm \lambda\sigma_z)} \\ &= \frac{1}{2}(\mathbb{I} \pm \lambda\sigma_z).\end{aligned} \tag{19}$$

The probability of getting '±' outcomes are both 1/2. In order to formulate the relevant LGI, the $\rho_\pm^{PM}(t_1)$ is evolved for time Δt, giving $\exp^{-i\frac{\omega\Delta t}{2}\sigma_x} \frac{1}{2}(\mathbb{I} \pm \lambda\sigma_z) \exp^{i\frac{\omega\Delta t}{2}\sigma_x} = \frac{\mathbb{I}}{2} \pm \frac{\lambda}{2}(\cos(\omega\Delta t)\sigma_z + \sin(\omega\Delta t)\sigma_y)$. With this post-measurement state we find the conditional probabilities and the two-time correlation function given by $C_{12} = \lambda^2 \cos(\omega\Delta t)$. Hence, the LGI with unsharp measurement can be written as $K \equiv \lambda^2\langle LGI\rangle \leq 2$, where $\langle LGI\rangle$ denotes the corresponding expression for sharp measurements. Since $\langle LGI\rangle_{max} = 2\sqrt{2}$, hence it follows that in the case of unsharp measurements the LGI for a spin 1/2 system is always satisfied when the sharpness parameter upper bounded by $\lambda < 1/2^{1/4}$.

Now, let us consider a system having arbitrary spin. As discussed in our conceptual scheme of measurement in the previous section, particles of spin j are sent from the source to an inhomogeneous electric field. After emerging from the field each beam is effectively described by a two dimensional Hilbert space, and evolves under the same unitary as above. Finally, the beam is subjected to a non-ideal Stern-Gerlach apparatus [31]. In this scenario the effective spin j observable is given by $Q = \lambda((\Gamma_z + \Pi)/\sqrt{2j+1})$, where $0 < \lambda \leq 1$. Using the lemma in section III it is straightforward to calculate the two-time correlation function. When N is even, we have

$$C_{12} = \lambda^2[\cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2] \tag{20}$$

and for N odd, it becomes

$$C_{12} = \lambda^2[2j\cos\alpha_1\cos\alpha_2 + 2j\sin\alpha_1\sin\alpha_2 + 1/\sqrt{2}]/(2j+1) \tag{21}$$

For both the even and odd cases the optimal value of the sharpness parameter below which no quantum violation of LGI is possible is upper bounded by $1/2^{1/4} \equiv 0.841$. Note in comparison that using the maximal violation of the LGI for large spin obtained in Ref. [15], the required value of this parameter would be 0.8978 in order to ensure satisfaction of the LGI. Note also, that for the case of spatial correlations, a higher value of the sharpness parameter would be required to ensure satisfaction of the relevant local realist inequality, since the maximal bound there drops for the case of integral spin [9]. For the case of spin-1 particles the value of the required sharpness for spatial correlations turns out to be 0.8852 and coincides with our optimal value (0.841) for temporal correlations with infinitely large integral spin.

V. LG-CH INEQUALITIES AND EMERGENCE OF CLASSICALITY

We finally show how satisfaction of the LGI implies that there exists a non-invasive realist model for the temporal correlations as described in section II. Following the line of reasoning used by Fine [27] for the case of spatial correlations, we show here that satisfaction of the LGI inequality is a necessary and sufficient condition for the existence of such a classical model in a situation when no-signaling in time (NSIT) [17] is satisfied. Fine's theorem [27] showed that Bell-CH inequalities provide a necessary and sufficient condition for existence of joint probabilities of all measured observables implying a local realist model. Recently, a new proof of Fine's theorem has been presented [33]. The similar nature of the local hidden variable models in the spatial scenarios and the non-invasive hidden variable models in the temporal scenario [23] enables the adoption of Fine's theorem in the temporal domain. To this end we first derive a CH type [32] LGI, and make clear how satisfaction of such LGI-CH implies existence of a non-invasive realist model (NIRM).

We assume the following properties of the joint and single probabilities, $p^{++}(Q_1Q_2)+p^{+-}(Q_1Q_2) = p^{+}(Q_1)$, $p^{++}(Q_1Q_2) + p^{-+}(Q_1Q_2) = p^{+}(Q_2)$ and $p^{+}(Q_1) + p^{-}(Q_1) = 1$, $p^{+}(Q_2) + p^{-}(Q_2) = 1$. Now the two-time correlation function becomes $C_{12} = 4p^{++}(Q_1, Q_2) - 2p^{+}(Q_1) - 2p^{+}(Q_2) + 1$. It is to be noted that obtaining single probabilities from joint probabilities amounts to satisfying the assumption of induction at the statistical level and no-signaling in time (NSIT). Unlike in the Bell scenario where the no-signaling principle holds, temporal correlation can violate NSIT. However, induction is always satisfied. It is shown in Ref. [17] that NSIT is an alternative necessary condition of MR.

The normalization condition for the single probabilities implies that ideal lossless detectors are used for measuring outcomes. Writing the two-time correlation in this form one can derive an inequality in a form equivalent to the Bell-CH inequality [32] in the temporal domain. We call such an inequality as an LGI-CH inequality, given by

$$\begin{aligned} -1 \leq p^{++}(Q_1Q_2) + p^{++}(Q_3Q_2) - p^{++}(Q_1Q_4) \\ +p^{++}(Q_3Q_4) - p^{+}(Q_3) - p^{+}(Q_2) \leq 0. \end{aligned} \quad (22)$$

Under the above conditions other LGI-CH inequalities can be derived by varying outcome combinations. These LGI-CHs may be combined to obtain the LGI. Now, it is straightforward following the line of resoning presented by Fine [27] that satisfaction of these inequalities imply the existence of joint probability distributions for all the observables, i.e., $p(Q_1, Q_2, Q_3, Q_4)$. This in turn implies a noninvasive realist model that mimics the temporal correlations for the measurement of a dichotomic observable at time t_1, t_2, t_3, t_4. Such a model is compatible with classical theory, and the quantum dynamics of the system boils down to some classical stochastic process with the measurement statistics given by averaging over this process. In precise we prove that

$$LGI \wedge NSIT <==> LG - CH <==> NIRM \quad (23)$$

VI. CONCLUSIONS

To summarize, in the present work we have shown that optimal violation of the Leggett-Garg inequality [13, 14] is allowed by quantum theory in the context of a suitably adopted measurement scheme for a system possessing arbitrary spin. The observable [9] chosen here enables one to achieve the maximal limit of temporal correlations irrespective of the integral or half-integral value of the spin, improving upon earlier results obtained through the choice of other observables [15]. It may be noted that whereas we obtain maximal violation of macrorealism for an arbitrary spin system, the same Peres-Gisin observable [9] used in the case of spatial correlations does not lead to maximal violation of the corresponding local realist inequality for finite integral spin systems.

We have further shown how coarse graining of the measurement process through unsharp observables [25] leads to the satisfaction of LGI. The form of coarse graining used here is quantitative, as different from the coarse-graining employed in a similar context earlier [15]. Here it is possible to obtain the precise threshold value of the sharpness parameter below which no quantum violation of the LGI can be achieved. Our approach using unsharp measurements fits naturally within the context of non-ideal apparatus [31] in actual experimental conditions. Comparing with the case of spatial correlations for a similar coarse grained approach through unsharp measurement, we find here that for temporal correlations the satisfaction of LGI emerges below a smaller value of the sharpness parameter. This lends further credence to the contention [23] of temporal correlations being somehow 'stronger' than spatial correlations. We finally show that satisfaction of the the LGI implies existence of a non-invasive realist model for dichotomic measurement at four different times in a situation when NSIT is satisfied.

Note : Recently, Clemente and Kofler have shown in Ref.[34, 35] how various (more than two time) NSIT and Arrow-of-Time (induction) form a necessary and sufficient condition for MR [36].

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